St Stephen's School – Carramar Campus
Mathematics Specialist 3AMAS **Investigation 2: Validation**
Total Marks: 50 **Time Allowed: 60 mins** **Resource Rich**

Name: _____ **Mark:**

Teacher:_____

Parent Signature:

INSTRUCTIONS

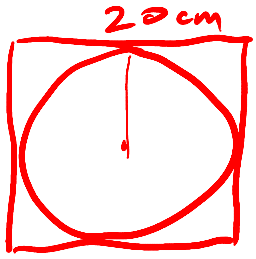
Permitted equipment:

- Two calculators complying with Curriculum Council requirements
- Both sides of one A4 page of notes
- Stationery and drawing equipment

Part A

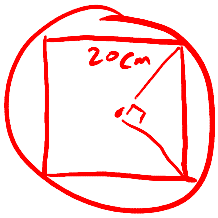
Question 1. **[7 marks]**

(a) A circle is inscribed in a square of side 20 cm in such a way that the circle touches all 4 sides of the square. What percentage of the area of the square lies outside the circle? (3 marks)



$A_{\square} = 20^2$
$= 400\ cm^2$
$r = 10\ cm$ ✓
$A_{\circ} = \pi r^2$
$= 100\pi$
Percent outside circle $= \frac{400 - 100\pi}{400} \times 100\%$ ✓
$= 21.46\%$ ✓

(b) A square is inscribed in a circle of radius 20 cm in such a way that all 4 vertices of the square lie on the circumference of the circle. What percentage of the area of the circle lies outside the square? (4 marks)



side length $\ell = \sqrt{20^2 + 20^2}$
$= 20\sqrt{2}$ ✓
square area $= \ell^2$
$= 800\ cm^2$ ✓
circle area $= \pi \times 20^2$
$= 400\pi$
percent outside square $= \frac{400\pi - 800}{400\pi}$ ✓
$= 36.34\%$ ✓

Question 2. [12 marks]

Note: The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$

(a) A sphere is inscribed in a cube of side 20 cm in such a way that the sphere touches all 6 faces of the cube. What percentage of the volume of the cube lies outside the sphere?

(4 marks)

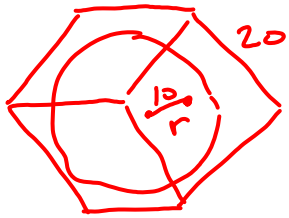
$r = 10$ cm ✓

$V_{cube} = 20^3$

$= 8000\ cm^3$

$V_{sphere} = \frac{4}{3}\pi r^3$

$= \frac{4}{3}\pi \times 10^3$ ✓



percent outside sphere $= \dfrac{8000 - \frac{4000\pi}{3}}{8000} \times 100\%$ ✓

$= 47.64\%$ ✓

(b) A cube is inscribed in a sphere of radius 20 cm in such a way that all 6 vertices of the cube lie on the surface of the sphere.

(i) Show clearly that the length of one edge of the cube is $\frac{2\sqrt{3}}{3} \times 20$ cm.

(4 marks)

the diagonal of the cube is the diameter : 40 cm ✓

Let l be the side length

$AC = \sqrt{l^2 + l^2} = l\sqrt{2}$

$BC = l$

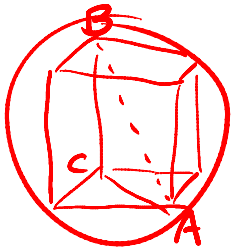
$AB = \sqrt{(l^2 + l^2) + l^2}$ ✓✓

$= l\sqrt{3}$

$l\sqrt{3} = 40$

$l = \frac{40}{\sqrt{3}}$ ✓

$= \frac{40\sqrt{3}}{3} = \frac{2\sqrt{3}}{3} \times 20$ cm



(ii) What percentage of the volume of the sphere lies outside the cube? (4 marks)

$V_{cube} = \left(\frac{2\sqrt{3}}{3} \times 20\right)^3$ ✓

$= \frac{8 \times 3\sqrt{3} \times 8000}{27 \cdot 9}$

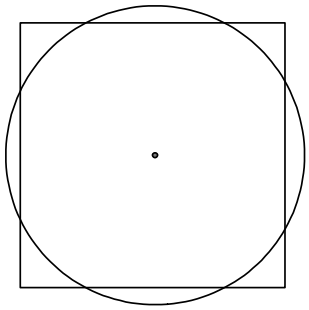
$= \frac{64000\sqrt{3}}{9}$ ✓

$pc. = \frac{\frac{4}{3}\pi \times 20^3 - \left(\frac{2\sqrt{3}}{3} \times 20\right)^3}{\frac{4}{3}\pi \times 20^3} \times 100\% =$ ✓

$= 63.24\%$ ✓

Question 3. [15 marks]

A circle and a square have the same area, as shown in the diagram. The radius of the circle is 20 cm.

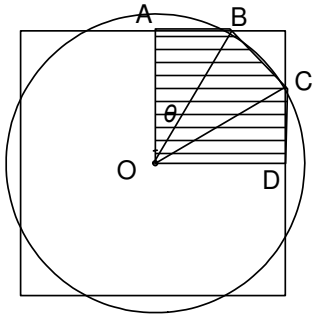


(a) Show clearly that the length of one side of the square is $20\sqrt{\pi}$ cm (1 mark)

$\pi \times 20^2 = l^2$

$l = 20\sqrt{\pi}$ cm ✓

We wish to calculate the area that is common to the square and the circle. The symmetry of the diagram allows us to consider one quarter of the common area as shown right.



(b) Find the length of OA. (1 mark)

$OA = l/2$

$= 10\sqrt{\pi}$

$(\approx 17.72$ cm$)$

(c) Calculate the size of angle θ in radians. (2 marks)

$\cos\theta = \frac{OA}{OB}$

$= \frac{17.72}{20}$

$\theta = \cos^{-1}\frac{17.72}{20}$

$= 0.4817$ RAD.

(d) Use the result from **(c)** to determine the area of the sector OBC. (4 marks)

$A = \frac{1}{2} r^2 \left(\frac{\pi}{2} - 2\theta\right)$ ✓✓

$= \frac{1}{2} \times 20^2 \left(\frac{\pi}{2} - 2 \times 0.4817\right)$ ✓

$= 121.50 \text{ cm}^2$ ✓

(e) (i) Calculate the length of AB. (2 marks)

$AB = \sqrt{20^2 - 10^2\pi}$ ✓

$= 9.27$ ✓

$10\sqrt{\pi}$ $\sqrt{20}$

(ii) Calculate the area of ΔOAB. (2 marks)

$\text{Area} = \frac{1}{2} \times 10\sqrt{\pi} \times 9.27$ ✓

$= 82.11 \text{ cm}^2$ ✓

(f) Explain why the two triangles, ΔOAB and ΔOCD, have the same area. (1 mark)

$\Delta ODC \cong \Delta OAB$ – symmetry of the square ✓

(g) What is the area common to the square and the circle? (2 mark)

$A = 4\left(121.50 + 2 \times 82.11\right)$ ✓

$= 1142.85 \text{ cm}^2$ ✓

Part B

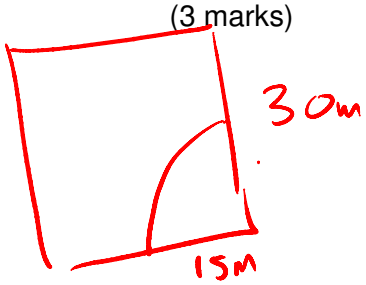
Question 1 **(6 marks)**

A farmer uses a rope to tether a goat to a pole at the corner of a square paddock. The length of one side of the square is 30 m.

(a) If the length of the rope is 15 m, what percentage of the paddock can the goat reach for grazing? Justify your answer. (3 marks)

$\frac{\frac{1}{4}\pi \times 15^2}{30^2} \times 100\%$ ✓ ✓

$= 19.63\%$ ✓



(b) If the farmer wants the goat to be able to graze exactly half of the area of the paddock, what length must the rope be? Justify your answer and record the length correct to 2 decimal places. (2 marks)

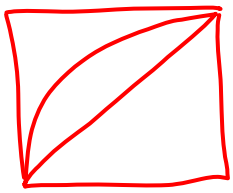
$\frac{1}{4}\pi \times r^2 = \frac{1}{2} \times 30^2$ ✓

$r = \sqrt{\frac{2 \times 30^2}{\pi}} = 30\sqrt{\frac{2}{\pi}}$

$\approx 23.94\text{m}$ ✓

(c) Give a mathematical reason to explain why the length of the rope in part (b) must always be less that the length of the side of the square paddock, regardless of what that length might be. (1 mark)

Because the sector is greater in area than the triangle. ✓



Question 2 **(10 marks)**

The farmer moves the goat to another paddock with more feed on it. This paddock is rectangular, with length 30 metres and width 15 metres. The goat is again tethered in one corner of this new paddock.

(a) Show that a rope of length 15 metres will not allow the goat to access half of the paddock. (3 marks)

Area accessed $= \frac{1}{4}\pi \times 15^2$ ✓

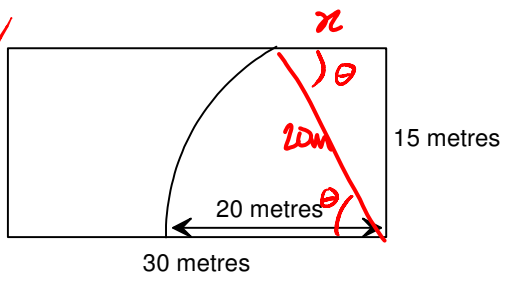
Area paddock $= 15 \times 30$

$= 2 \times 15^2$

Half paddock $= 15^2$ ✓

$\frac{1}{4}\pi \times 15^2 < 15^2$ ✓ (since $\frac{\pi}{4} < 1$)

(b) Calculate the percentage of the paddock that the goat can graze over if the rope is 20 metres long, as illustrated in the diagram below. (7 marks)



$x = \sqrt{20^2 - 15^2}$

$= 13.23\text{ m}$ ✓

$\sin\theta = \frac{15}{20}$

$\theta = \sin^{-1} 0.75$

$= 0.8481$ ✓

$A = \frac{1}{2} r^2\theta + \frac{1}{2} \times 15x$

$= \frac{1}{2} \times 20^2 \times 0.8481 + \frac{1}{2} \times 15 \times 13.23$ ✓

$= 169.61$ ✓ $+ 99.22$ ✓

$= 268.83\text{ m}^2$ ✓

$\text{Percentage} = \frac{268.83}{15 \times 30} \times 100\%$

$= 59.74\%$ ✓